The Kolmogorov turbulence theory in the light of six-dimensional Navier-Stokes' equation

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Abstract. The classical turbulence theory by Kolmogorov is reconsidered using Navier-Stokes' equation generalized to 6D physical-plus-eddy space. Strong pseudo-singularity is shown to reveal itself along the boundary 'ridge' line separating the dissipation and inertial sub-ranges surrounding the origin of the eddy space. A speculation is made that this singularity is generated by two dipoles of opposite sign aligned on the common axis. It is supported by the observation that the universal power spectrum calculated rediscovers the Kolmogorov's -5/3 power law as independent of the dimensional approach.

1. Introduction

As early as in 1941 Kolmogorov[1] predicted some universal features of fluid turbulence which have been confirmed by experiments in later years. It is rather surprising that they are derived from dimensional analysis based on the simple assumption of 'local homogeneity' for small-scale turbulence. More surprising is the fact that those have little to do with the equation of dynamics of fluid.

It is addressed in this paper to locate this 'missing link' through rederivation of the -5/3 power law of the spectrum for the inertial subrange as the universal law using the dynamical equation proposed in ref.2. The key issue for this task to work is that equation on which to describe the features of small eddies as independent of individual flow geometry. To meet this purpose a six-dimensional Navier-Stokes equation is employed, where additional 3D space having a length dimension , corresponding to eddy size, is introduced.

Originally such an equation has been derived elsewhere[2], using non-equilibrium statistical mechanics starting from Liouville's equation. In this formalism the 3D 'eddy' space has been introduced as natural consequence of a mathematical procedure, namely, the separation of variables of turbulent fluctuation-correlation equation. The equation stands as 6D generalization of the Navier-Stokes equation which in 3D physical space degenerates to the classical equation.

In what follows, however, it is intended to rederive the same equation using phenomenologies alone on the basis same as the classical theory of Kármán and Howarth[3]. Then, the wave-number space is introduced for separating variables

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(Sec.2). It is then Fourier-transformed into eddy space, thereby the 6D Navier-Stokes equation, the basis to all what follows is established (Sec.3). The equation gives a novel expression for turbulent dissipation, enabling to predict existence of a pseudo-singularity surrounding the dissipation region (Sec.4). In inertial subrange, this singularity turns out to be a couple of dipoles of opposite sense, separated by the order of several tens to hundreds the Kolmogorov length (Sec.5). Power spectrum of this locally homogeneous turbulence is calculated, showing a wave-number dependence close to -5/3 power law of the Kolmogorov theory.(Sec.6)

2. Kármán-Howarth formalism revisited as governing inhomogeneous turbulence

In 1938, von Kármán and Howarth proposed an equation governing homogeneous isotropic turbulence whose original form is written as[3]

$$\overline{u'_j(\widehat{\mathrm{NS}})_l + \hat{u}'_l(\mathrm{NS})_j} = 0 \tag{1}$$

In this equation $u'_j = u'_j(\boldsymbol{x}, t)$ and $\hat{u}'_l = u'_l(\widehat{\boldsymbol{x}}, t)$ are instantaneous velocity fluctuation at $\boldsymbol{x}$ and $\widehat{\boldsymbol{x}}$, respectively, overbar ($\overline{}$) denotes the conventional (ensemble) average and

$$\begin{aligned}\mathbf{NS} &\equiv \mathbf{NS}(\nabla, \underline{\boldsymbol{u}}, \underline{p}) \\ &= (\frac{\partial}{\partial t} + \underline{\boldsymbol{u}} \cdot \nabla - \nu\nabla^2)\underline{\boldsymbol{u}} + \rho^{-1}\nabla\underline{p} = 0\end{aligned} \tag{2}$$

denotes the Navier-Stokes equation written in terms of the instantaneous fluid quantities

$$\begin{aligned}\underline{\boldsymbol{u}} &= \boldsymbol{u} + \boldsymbol{u}' \\ \underline{p} &= p + p'\end{aligned} \tag{3}$$

where $\boldsymbol{u}$ and p are the average velocity and pressure, respectively, ν is the kinematic viscosity, and ∇ denotes nabla vector. Likewise $\widehat{\mathbf{NS}}$ is defined by Eq.(2) in which ($\nabla, \underline{\boldsymbol{u}}, \underline{p}$) is replaced with ($\hat{\nabla}, \underline{\hat{\boldsymbol{u}}}, \underline{\hat{p}}$).

Eq.(1) is an equation in 6-D physical space and time, namely, has seven independent variables $(\boldsymbol{x}, \hat{\boldsymbol{x}}, t)$. The classical theory[3] chose to discuss homogeneous and isotropic turbulence to reduce independent variables to $(|\boldsymbol{x} - \hat{\boldsymbol{x}}|, t)$. We can show, however, that the equation can be renovated to be available for inhomogeneous turbulence as well by introducing separation of variables, together with a closure condition which seems plausible intuitively.

The actual derivation proceeds as follows: Eq.(1) in which $\mathbf{NS}(\nabla, \underline{\boldsymbol{u}}, \underline{p})$ is decomposed into average and fluctuating parts using (3),reads

$$\overline{u'_j(\widehat{\mathrm{NS}})'_l + \hat{u}'_l(\mathrm{NS})'_j} = 0 \tag{4}$$

where

$$(\mathrm{NS})'_j \equiv (\frac{\partial}{\partial t} + u_r\frac{\partial}{\partial x_r} - \nu\frac{\partial^2}{\partial x_r^2})u'_j + \frac{1}{\rho}\frac{\partial p'}{\partial x_j} + \frac{\partial}{\partial x_r}(u'_j u'_r) \tag{5}$$

together with $(\widehat{\mathrm{NS}})'_l$ defined as in Eq.(1). Upon substituting Eq.(5) into Eq.(4) we see that the equation consists of terms of double and triple fluctuation correlations, which we will decompose as

$$\overline{u'_j\hat{u}'_\lambda} = \mathrm{R.P.}l^6\int_{-\infty}^{\infty} g_j(\boldsymbol{x},\boldsymbol{k})g_\lambda(\hat{\boldsymbol{x}},\hat{\boldsymbol{k}})\delta(\boldsymbol{k}+\hat{\boldsymbol{k}})d\boldsymbol{k}d\hat{\boldsymbol{k}}\ ,\ \lambda=(l,4) \tag{6}$$

$$\overline{u'_j\hat{u}'_l\tilde{u}'_r} = \mathrm{R.P.}l^9\int_{-\infty}^{\infty} g_j(\boldsymbol{x},\boldsymbol{k})g_l(\hat{\boldsymbol{x}},\hat{\boldsymbol{k}})g_r(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{k}})\delta(\boldsymbol{k}+\hat{\boldsymbol{k}}+\tilde{\boldsymbol{k}})d\boldsymbol{k}d\hat{\boldsymbol{k}}d\tilde{\boldsymbol{k}} \tag{7}$$

where $\boldsymbol{k}$ is the wave number, l is the characteristic length of flow geometry, R.P. denotes taking real part, and subscript λ runs 1 through 4. A supplementary definition

$$u'_4 = \rho^{-1}p' \tag{8}$$

stands for the pressure fluctuation. It is more convenient for later use to employ the following alternative expressions:

$$\overline{u'_j\hat{u}'_\lambda} = \mathrm{R.P.}l^3\int_{-\infty}^{\infty} g_j(\boldsymbol{x},\boldsymbol{k})g_\lambda(\hat{\boldsymbol{x}},-\boldsymbol{k})d\boldsymbol{k}\ ,\ \lambda=(l,4) \tag{6'}$$

$$\overline{u'_j\hat{u}'_l\tilde{u}'_r} = \mathrm{R.P.}l^6\int_{-\infty}^{\infty} g_j(\boldsymbol{x},\boldsymbol{k})d\boldsymbol{k}\int_{-\infty}^{\infty} g_l(\hat{\boldsymbol{x}},-\boldsymbol{k}+\hat{\boldsymbol{k}})g_r(\tilde{\boldsymbol{x}},-\hat{\boldsymbol{k}})d\hat{\boldsymbol{k}}\ ,\ (\tilde{\boldsymbol{x}};\boldsymbol{x}\ \text{or}\ \hat{\boldsymbol{x}}) \tag{7'}$$

It is to be noted that there is no a priori reason why the double and triple correlations are to be expressed in terms of the *same* g_λ. At this point it is simply invoked as the closure condition to truncate the chain of equations at the level of Eq.(4). This closure can be justified only through the quality of outcome as to whether it fits with physical reality of turbulence.

With these preliminaries, Eq.(4) with assumption (6',7') substituted into, leads to the form typical of separation of variables, here into ($\boldsymbol{x}$,t) and ($\hat{\boldsymbol{x}}$,t), respectively;

$$\int_{-\infty}^{\infty} d\boldsymbol{k}g_j\hat{g}_l[\underbrace{g_j^{-1}(\mathrm{ns})_j^{(0)}}_{=\ i\omega}+\underbrace{\hat{g}_l^{-1}(\widehat{\mathrm{ns}})_l^{(0)}}_{=\ -i\omega}]=0 \tag{9}$$

where ω is the separation parameter having the dimension of the frequency, and $\hat{g}_l$ stands for $g_l(\hat{\boldsymbol{x}},\boldsymbol{k})$, and $(\mathrm{ns})^{(0)}$ is defined by

$$(\mathrm{ns})_j^{(0)} \equiv [\mathrm{ns}(\boldsymbol{g},g_4)^{(0)}]_j = (\frac{\partial}{\partial t}+u_r\frac{\partial}{\partial x_r}-\nu\frac{\partial^2}{\partial x_r^2})g_j+\frac{\partial u_j}{\partial x_r}g_r+\frac{\partial g_4}{\partial x_j}+\frac{\partial}{\partial x_r}I(g_jg_r) \tag{10}$$

with

$$I(g_jg_r) \equiv l^3\int_{-\infty}^{\infty} g_j(\boldsymbol{k}-\hat{\boldsymbol{k}})g_r(\hat{\boldsymbol{k}})d\hat{\boldsymbol{k}}$$

Imaginary factor i as appearing in Eq.(9) reflects the statistical symmetry of the tensor $\overline{u'_j\hat{u}'_l}$, which is met under the following additional condition

$$g_l(\hat{\boldsymbol{x}},-\boldsymbol{k}) = [g_l(\hat{\boldsymbol{x}},\boldsymbol{k})]^* \tag{11}$$

In fact, then, the two terms inside[ ] of integral (9) are commutable to each other through taking complex conjugate(*), if $i\omega$ is purely imaginary. Thus we are led to the assertion that the only equation that need to be solved is

$$i\omega g_j = (\mathrm{ns})_j^{(0)} \tag{12}$$

To go further from this point on, we need to have relationship between frequency ω introduced as the separation parameter and wave number $\boldsymbol{k}$ by which the solution is constructed in the form of bilinear integral. Physically they are related to each other by the dispersion relation $\omega(\boldsymbol{k})$. Or, instead, we may introduce phase velocity $\boldsymbol{c}$ by

$$\omega = \boldsymbol{c} \cdot \boldsymbol{k} \tag{13}$$

with no loss of generality. In fact, its constancy does not mean the turbulent eddies being assumed as nondispersive. For the unsteady term in the equation ($\partial/\partial t \neq 0$) is responsible for the dispersive part, if any.

Owing to the convolution form of integral (10) periodic factor drops off from Eq.(12) despite its nonlinear structure by putting

$$g_\lambda(\boldsymbol{x}, \boldsymbol{k}) = e^{i\boldsymbol{k}\cdot\boldsymbol{x}} f_\lambda(\boldsymbol{x}, \boldsymbol{k}) \ , \ \lambda = (l, 4) \tag{14}$$

thereby we have the following equation governing its amplitude f_λ,

$$i\omega f_j = (\text{ns})_j \tag{15}$$

where

$$\begin{aligned}(\text{ns})_j &\equiv [\text{ns}(\boldsymbol{f}, f_4)]_j \\ &= [\frac{\partial}{\partial t} + u_r \partial_r(\boldsymbol{k}) - \nu \partial_r^2(\boldsymbol{k})] f_j + \frac{\partial u_j}{\partial x_r} f_r + \partial_j(\boldsymbol{k}) f_4 + \partial_r(\boldsymbol{k}) I(f_j f_r)\end{aligned} \tag{16}$$

with

$$\boldsymbol{\partial}(\boldsymbol{k}) \equiv \nabla(\boldsymbol{x}) + i\boldsymbol{k} \tag{17}$$

It is readily checked that expressions defined by (15) and (10) are related to each other by

$$(\text{ns})_j^0 = [(\text{ns})_j]_{\boldsymbol{\partial}(\boldsymbol{k}) \to \nabla(\boldsymbol{x})}$$

Several remarks are in order with regards to physical implications of variables having appeared in this section: Amplitude function $\boldsymbol{f}$ as obeying Eq.(14) is *essentially* complex and is *not* an observable. It is related to the observable quantity of fluctuation correlation through

$$\overline{u_j' \hat{u}_l'} = \text{R.P.} l^3 \int_{-\infty}^{\infty} e^{i\boldsymbol{k}\cdot(\boldsymbol{x} - \hat{\boldsymbol{x}})} f_j(\boldsymbol{x}, \boldsymbol{k}) f_l^*(\hat{\boldsymbol{x}}, \boldsymbol{k}) d\boldsymbol{k} \tag{18}$$

as is confirmed by (6'),(11)and (13). Its complex variable structure has its origin in Eq.(9) where the imaginary unit i is introduced from the symmetry postulate of statistical mechanics[2] for correlation tensor (18). We may note some coincidental parallelism to Schroedinger's wave equation where the imaginary factor secures the corresponding tensor to be Hermitean as it should.

The most crucial on which this paper rests is the soundness of the basis of Eq.(1). Originally it was a product of intuition by Theodore von Kármán without any 'first principle' ground shown in the paper[3]. In fact, then, any linear combinations of $(\mathbf{NS})'$ would be claimed as equally qualified. Firm basis is provided by nonequilibrium statistical mechanics[4] to justify that the linear combination only of the form (1) is consistent with Liouville's equation, namely, the equation of continuity in the phase space of Hamiltonian mechanics. .

3. Turbulence in eddy space

The nonlinear integro-differential equation we have derived in the previous section is not very easy to deal with. Since, however, the integral is of convolution type, it transforms to a simple product, through Fourier transform as

$$f_j(\boldsymbol{x},\boldsymbol{k}) = (2\pi l)^{-3}\int_{-\infty}^{\infty} d\boldsymbol{s} e^{-i\boldsymbol{k}\cdot\boldsymbol{s}} q_j(\boldsymbol{x},\boldsymbol{s}) \tag{19}$$

The actual operation of the transform on Eq.(14) gives

$$[-c_r\frac{\partial}{\partial s_r}+\frac{\partial}{\partial t}+u_r\partial_r(\boldsymbol{s})-\nu\partial_r^2(\boldsymbol{s})]q_j+\partial_j(\boldsymbol{s})q_4+\frac{\partial u_j}{\partial x_r}q_r+\partial_r(\boldsymbol{s})(q_jq_r) = 0 \tag{20}$$

where

$$\partial_j(\boldsymbol{s}) \equiv \partial/\partial x_j + \partial/\partial s_j \tag{21}$$

Variable $\boldsymbol{s}$ introduced in Eq.(19) as the Fourier variable adjoint to wave number $\boldsymbol{k}$ has dimension of length, which stands for eddy size with direction of the vorticity vector, so may well be called *eddy variable*. It should be remarked that Eq.(20) is the Navier-Stokes equation in 6D (physical plus eddy) space, describing the motion of turbulent vortices moving with phase velocity $\boldsymbol{c}$. In fact, this equation governing $q_\lambda(\boldsymbol{x},\boldsymbol{s})$ for prescribed $\boldsymbol{u}(\boldsymbol{x})$ and $p(\boldsymbol{x})$ is alternatively written as

$$[\boldsymbol{c}\cdot\nabla(\boldsymbol{s})]\boldsymbol{q} = \mathbf{NS}(\boldsymbol{\partial}(\boldsymbol{s}),\boldsymbol{u}+\boldsymbol{q},p+\rho q_4) - \mathbf{NS}(\nabla(\boldsymbol{x}),\boldsymbol{u},p) \tag{22}$$

with

$$\boldsymbol{\partial}(\boldsymbol{s}) \equiv \nabla(\boldsymbol{x}) + \nabla(\boldsymbol{s}) \tag{23}$$

where $\mathbf{NS}$ has been defined by Eq.(2), and ∇ denote the nabla vector in the respective spaces§.

Simplicity in expression for physical quantities is another advantage of working with $\boldsymbol{s}$-space. For example, fluctuation-correlation formula (18) takes remarkably simple form

$$\overline{u'_j\hat{u}'_l} = (2\pi l)^{-3}\int_{-\infty}^{\infty} d\boldsymbol{s} q_j(\boldsymbol{x},\boldsymbol{s}+\boldsymbol{x})q_l(\hat{\boldsymbol{x}},\boldsymbol{s}+\hat{\boldsymbol{x}}) \tag{24}$$

as is easily confirmed by substituting (19) and employing definition of the delta function $\delta(\boldsymbol{s}/l) = (2\pi l)^3\int e^{i\boldsymbol{k}\cdot\boldsymbol{s}}d\boldsymbol{k}$. Turbulent dissipation which has been dealt with as an elementary parameter in the classical dimensional analysis can be expressed in this space by an explicit form: We have, by definition,

$$\begin{aligned}
\frac{\epsilon}{2} &= \frac{\nu}{2}\overline{\left(\frac{\partial u'_j}{\partial x_l}+\frac{\partial u'_l}{\partial x_j}\right)^2} \\
&= \frac{\nu}{2}\overline{\left[\left(\frac{\partial u'_j}{\partial x_l}+\frac{\partial u'_l}{\partial x_j}\right)\left(\frac{\partial \hat{u}'_j}{\partial \hat{x}_l}+\frac{\partial \hat{u}'_l}{\partial \hat{x}_j}\right)\right]}_{\hat{\boldsymbol{x}}=\boldsymbol{x}} \\
&= \nu\left[\left(\frac{\partial^2\overline{u'_j\hat{u}'_j}}{\partial x_l\partial\hat{x}_l}\right)+\left(\frac{\partial^2\overline{u'_j\hat{u}'_l}}{\partial x_l\partial\hat{x}_j}\right)\right]_{\hat{\boldsymbol{x}}=\boldsymbol{x}}
\end{aligned} \tag{25}$$

§ The bold-face letters denote vectors having three components, not to be confused with 6D vectors.

On the other hand, we have from (24)

$$\frac{\partial^2 \overline{u_j' \hat{u}_l'}}{\partial x_m \partial \hat{x}_n} = \frac{1}{(2\pi l)^3} \int_{-\infty}^{\infty} d\boldsymbol{s} \left(\frac{\partial q_j}{\partial x_m} + \frac{\partial q_j}{\partial s_m} \right) \left(\frac{\partial \hat{q}_l}{\partial \hat{x}_n} + \frac{\partial \hat{q}_l}{\partial \hat{s}_n} \right)_{\hat{\boldsymbol{s}}=\boldsymbol{s}}$$

$$= \frac{1}{(2\pi l)^3} \int_{-\infty}^{\infty} d\boldsymbol{s} [\partial_m(\boldsymbol{s}) q_j][\hat{\partial}_n(\hat{\boldsymbol{s}}) \hat{q}_l]_{\hat{\boldsymbol{s}}=\boldsymbol{s}}$$

Thus we are led to the final expression for dissipation ϵ as

$$\frac{\epsilon}{2} = \frac{\nu}{(2\pi l)^3} \int_{-\infty}^{\infty} d\boldsymbol{s} [(\partial_l(\boldsymbol{s}) q_j)^2 + (\partial_l(\boldsymbol{s}) q_j)(\partial_j(\boldsymbol{s}) q_l)] \tag{26}$$

This formula will reveal a new facet of the dissipation function hidden in the classical working space.

4. Dimensional consideration on the existence of pseudo-singularity

In the Kolmogorov regime, dissipation ϵ was introduced, together with kinematic viscosity ν , characteristic velocity v and length l , respectively, as an elementary parameter by which to construct the dimensional analysis. They are related to each other through the following relationships

$$\epsilon \sim v^3/l \sim v_{\mathrm{K}}^3/l_{\mathrm{K}} \tag{27}$$

where the subscript K denotes the respective quantities in the dissipation region. The following formulae are their immediate consequences

$$\left.\begin{array}{rcll} v_{\mathrm{K}} l_{\mathrm{K}} &=& \nu & \\ v_{\mathrm{K}} &=& v R^{-1/4} & \\ l_{\mathrm{K}} &=& l R^{-3/4} & \quad \text{(Kolmogorov length)} \end{array}\right\} \tag{28}$$

where R is the flow Reynolds number

$$R = vl/\nu \tag{29}$$

Formula (26) we have derived in the newly defined hyperspace sheds some lights on this classical theory that is conducted within the physical space, in the sense that ϵ is not necessarily an elementary parameter any longer. According to Kolmogorov[1] viscous dissipation occurs only within the scale of Kolmogorov length. Then, the integral region of dissipation function (26) which is proportional to the kinematic viscosity is confined within a small volume of $O(l_{\mathrm{K}}^3)$. Dimensional analysis of (26) gives

$$\epsilon \sim \frac{\nu}{l^3} \left(\frac{\boldsymbol{q}}{l_{\mathrm{K}}} \right)^2 l_{\mathrm{K}}^3 \tag{30}$$

Thus, from (27) through (30) we are given for the order of magnitude of $|\boldsymbol{q}|$:

$$|\boldsymbol{q}|/v \sim R^{7/8} \tag{31}$$

It is an indicative of strong in/out mass flow existent (cf. the second of Eq.(28) of the classical theory) in the vicinity of this extremely small 'energy black hole' where the flow loses kinetic energy ϵ converted into heat. Inside this region $|\boldsymbol{s}| < O(l_{\mathrm{K}})$ the turbulence dies off rapidly towards $q(0) = 0$ .There must be, therefore, a drastic variation in the magnitude of $|\boldsymbol{q}|$ which peaks at the boundary ridgeline between dissipation and inertial ranges surrounding the origin ($\boldsymbol{s}$=0), and diminishes quickly inside.

In the neighborhood of this 'pseudo'-singularity it is obvious for the following conditions to hold:

$$\begin{aligned} |\boldsymbol{q}| &>> v \\ \underbrace{\partial/\partial s_j}_{O(l_{\mathrm{K}}^{-1})} &>> \underbrace{\partial/\partial x_j}_{O(l^{-1})} \end{aligned} \tag{32}$$

The latter condition warrants for local homogeneity in the sense of Kolmogorov to hold most strictly in the localized eddy space for any inhomogeneous turbulence. Under these circumstances, the 6D Navier-Stokes equation (20) together with equation of continuity reduce to

$$\partial q_j/\partial s_j = 0 \tag{33}$$

$$\left(q_r \frac{\partial}{\partial s_r} - \nu \frac{\partial^2}{\partial s_r^2} \right) q_j + \frac{\partial q_4}{\partial s_j} = 0 \tag{34}$$

which is nothing but the classical (3D) equations for laminar viscous flows, as disguised through

$$\boldsymbol{x} \rightarrow \boldsymbol{s} \ , \ (\boldsymbol{u}, p) \rightarrow (\boldsymbol{q}, \rho q_4) \tag{35}$$

This rule reigns dissipation region as well as the adjacent region of inertial subrange surrounding it, where $\boldsymbol{q}$-function diminishes outward off the pseudo-singularity until flow inhomogeneity starts to make its appearance. The solution as such, to be pursued in this space, is 'universal', namely, to be valid even for any shear turbulence.

According to Kolmogorov[1], the inertial range is where no viscous effects are operating. So we may claim that the potential flow is prevailing there. This speculation is supported by the assertion: '*potential flow is the solution of the Navier-Stokes equation in the region where no solid boundary is existent.*' Now is the case with it, because no physical substances are intervening here in this space.

5. The pseudo-singularity in the inertial subrange

The pseudo-singularity, as viewed from the domain of the inertial range, looks as if a genuine singularity, whose actual form is now to be identified.

We start with checking if the local isotropy in the classical sense has physical reality. The isotropy assumption is equivalent to

$$q_j = s_j Q(s) \tag{36}$$

which is the alternative expression of Robertson's theorem[5]. Substitution of (36) into equation of continuity (33) gives

$$\begin{aligned} s dQ/ds + 3Q = 0 \\ \therefore Q \sim s^{-3} \end{aligned}$$

Obviously, this is a source/ sink flow, corresponding to the velocity potential

$$\phi^{(0)} = \alpha s^{-1} \tag{37}$$

If α is positive it represents a sink flow with mass flux $4\pi\alpha$ vanishing at the origin. Reminding that the dissipation range $|\boldsymbol{s}| \lesssim l_{\rm K}$ is the 'black hole' of turbulent kinetic energy losing the amount by ϵ converted into heat every second, we see that no mass is supposed to be lost. Thus solution (36) for spherical isotropy is ruled out.

Prospective singularity now to take over can be sought within potential flow regime as follows: It is obvious that operator $\partial^N/\partial s_1^l \partial s_2^m \partial s_3^n (N = l + m + n)$ commutes with Laplacean operator, therefore

$$\phi^{(N)} = \frac{\partial^N \phi^{(0)}}{\partial s_1^l \partial s_2^m \partial s_3^n} \tag{38}$$

represent a group of potential flows. In particular

$$\phi_j^{(1)} = \frac{\partial \phi^{(0)}}{\partial s_j} \tag{39}$$

representing a dipole aligned with its axis parallel to s_j direction is seen to meet the purpose. In fact a dipole is made up of a pair of sink/source of equal strength, so no mass flux is lost at this spot. Thus simplest possible candidate for the pseudo-singularity is *axially* isotropic.

The following observation from direct numerical simulation[6] helps us draw a more precise picture of our pseudo-singularity: In the physical space the dissipation occurs only within a confined volume of 'elementary particles' of rod shape, randomly dispersed in turbulent medium. They all have a shape like a worm with radius σ_0 of several Kolmogorov lengths and several ten times of it lengthwise ($s_0 = N\sigma_0$), rotating round their own axes. Their lifetime is about $l_{\rm K}/v_{\rm K} = (l/v)R^{-1/2}$. So to say, they are like firefly worms illuminating light for their lifetime as small as of the order of far subseconds.

This picture when mapped onto our eddy space is such that two spinning dipoles having opposite sign and rotation are placed at

$$\boldsymbol{s} = (s_1, s_2, s_3) = (\pm s_0, 0, 0) \tag{40}$$

respectively, where the axis of symmetry is s_1 axis. The direction j = 1 is the direction of motion of turbulence-generating body, representing the only vector prescribing the fluid motion. The flow field induced by the pair of spinning dipoles

$$\left.\begin{array}{rcl} \boldsymbol{q} &=& \boldsymbol{q}_D + \boldsymbol{q}_S \\ \boldsymbol{q}_D &=& \boldsymbol{q}_D^+ + \boldsymbol{q}_D^- \\ \boldsymbol{q}_S &=& \boldsymbol{q}_S^+ + \boldsymbol{q}_S^- \end{array}\right\} \tag{41}$$

with $\boldsymbol{q}_D^\pm$ and $\boldsymbol{q}_S^\pm$ standing for velocities induced by dipoles and line vortices, placed at points (40), respectively:

$$\boldsymbol{q}_D^\pm = \pm\nabla(\boldsymbol{s})\frac{\partial}{\partial s_1}\frac{\alpha}{|\boldsymbol{s} \mp \boldsymbol{i}s_0|} \tag{42}$$

$$\boldsymbol{q}_S^{\pm} = \pm\frac{\beta}{4\pi}\delta(s_1 \mp s_0)\frac{\boldsymbol{i}\times(\boldsymbol{s}\mp\boldsymbol{i}s_0)}{|\boldsymbol{s}\mp\boldsymbol{i}s_0|^3} \tag{43}$$

where $\boldsymbol{i}$ is the unit vector designating s_1 axis. Expression (42) is direct consequence of (37) and (39), and that for (43) is Biot-Savart's law for a line vortex with infinitesimal length directing s_1-axis and with circulation β. For example, the actual form of $q_1(\boldsymbol{s})$ rewritten in axially isotropic form $q_1(s_1,\sigma)$ with $\sigma^2 = s_2^2 + s_3^2$ is,

$$\begin{aligned} q_1(s_1,\sigma) &= \frac{\partial^2}{\partial s_1^2}\left\{\frac{\alpha}{[(s_1-s_0)^2+\sigma^2]^{1/2}} - \frac{\alpha}{[(s_1+s_0)^2+\sigma^2]^{1/2}}\right\} \\ &= \alpha\left\{-\frac{2}{[(s_1-s_0)^2+\sigma^2]^{3/2}} + \frac{3\sigma^2}{[(s_1-s_0)^2+\sigma^2]^{5/2}}\right. \\ &\qquad \left. +\frac{2}{[(s_1+s_0)^2+\sigma^2]^{3/2}} - \frac{3\sigma^2}{[(s_1+s_0)^2+\sigma^2]^{5/2}}\right\} \end{aligned} \tag{44}$$

Note that there is no contribution from $\boldsymbol{q}_S$ to q_1. For $|s_1| >> s_0$ this expression approaches to

$$q_1 \sim 6\alpha s_0 s_1\left(-\frac{2}{s^5} + \frac{5\sigma^2}{s^7}\right) \tag{45}$$

which is a quadrupole field, as it should be expected.

Streamlines are shown in Fig.1 of the fictitious flow generated by a pair of dipoles given by the potential

$$\left.\begin{aligned} \phi^{(1)} &= \partial\phi^{(0)}/\partial s_1 \\ \phi^{(0)} &= \alpha\left\{[(s_1-s_0)^2+\sigma^2]^{-1/2} - [(s_1+s_0)^2+\sigma^2]^{-1/2}\right\} \end{aligned}\right\} \tag{46}$$

This axi-symmetric flow can also be represented using streamfuntion ψ as

$$\left.\begin{aligned} q_1 &= \partial\phi^{(1)}/\partial s_1 = \sigma^{-1}\partial(\sigma\psi)/\partial\sigma \\ q_\perp &= \partial\phi^{(1)}/\partial\sigma = -\sigma^{-1}\partial(\sigma\psi)/\partial s_1 \end{aligned}\right\} \tag{47}$$

from which we have

$$\psi = -\sigma\partial\phi^{(0)}/\partial\sigma \tag{48}$$

The flow pattern ψ :const. shows quadrupole-like structure at far field($|\boldsymbol{s}| >> s_0$) toward which the longitudinal vortices are stretched streamwise and then getting thicker. On the returning path to the dipole core they are chopped off and trim the aspect ratio, getting into the dissipation region. This picture may serve the qualitative description of what is actually observed (Fig.2).

Expression (45) gives us estimate for the outer boundary of the locally homogeneous region. That is also the inner boundary of inhomogeneous region where the first condition of inequality (32) ceases to hold;

$$|\boldsymbol{q}| \sim v \tag{49}$$

Let this boundary be defined by $s \sim l_0$, then we have $q \sim \alpha s_0 l_0^{-4}$ from (45), therefore condition (49) is replaced with a more precise one

$$v_0 \sim \alpha s_0 l_0^{-4} \tag{50}$$

where v_0 is the characteristic velocity corresponding to l_0. They supplement Kolmogorov formula (27) as

$$\epsilon \sim v_K^3/l_K \sim v_0^3/l_0 \sim v^3/l \tag{51}$$

On the opposite side of the inertial subrange $s - s_0 \sim l_{\rm K}$, $\sigma \sim l_{\rm K}$, q is estimated from (44) as

$$q(s_0) \sim \alpha l_{\rm K}^{-3}, \tag{52}$$

We note that the dimensional analysis developed in Sec.4 still holds by replacing $|\boldsymbol{q}|$ with $q(s_0)$ of (52), for instance,

$$q(s_0)/v \sim R^{7/8}. \tag{53}$$

Then, by eliminating $\alpha, q(s_0)$ and v_0 from (50) through (53) we have

$$l_0/l = (s_0/l)^{4/13} R^{-11/26} \tag{54}$$

The relationship between s_0 and $l_{\rm K}$ or l is yet to be reconsidered. At present no consensus formula is available for explicit parameter dependence of worm size s_0 . A recent observation by numerical experiments[8] is that s_0 is of the order of the Taylor microscale ($\sim l\,\overline{u'}/v$, $\overline{u'}$; r.m.s. of the velocity fluctuation), according to which the outer boundary of the inertial range is

$$l_0/l \sim R^{-11/26} (\overline{u'}/v)^{4/13} \tag{55}$$

6. Power spectrum for inertial subrange

The actual form of the pseudo-singularity as predicted in the preceding section enables us to calculate 1D power spectrum for 1D wave number k_1 in the inertial subrange. This spectrum function $P_{11}(k_1)$ is written, by definition,

$$\overline{u_1'^2} = \int_{-\infty}^{\infty} P_{11}(k_1) dk_1 \tag{56}$$

The well-known consequence of the classical dimensional analysis is that P_{11} , having the dimension of $v^2 l \sim \epsilon^{2/3} l^{5/3}$, reads in the language of wave number space as

$$P_{11} \sim k_1^{-5/3} \tag{57}$$

An alternative look into this formula on our own basis is the following: The actual form of P_{11} is obtained through comparing formula (18) for $\hat{\boldsymbol{x}} = \boldsymbol{x}$ and $j = l = 1$, namely,

$$\overline{u_1'^2} = \text{R.P.} l^3 \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 dk_3 f_1 f_1^*$$

with (56), which reads

$$P_{11}(k_1) = l^3 \int_{-\infty}^{\infty} dk_2 dk_3 f_1 f_1^* \tag{58}$$

This integral is, upon substitution of (19), transformed into the one in $\boldsymbol{s}$-space as

$$P_{11}(k_1) = (2\pi l)^{-3} \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{\infty} d\hat{s}_1 e^{-ik(s_1 - \hat{s}_1)} \int_0^{\infty} \sigma d\sigma q_1(s_1, \sigma) q_1(\hat{s}_1, \sigma) \tag{59}$$

Since the integration spans over the whole $\boldsymbol{s}$-space, we have yet to know solution q_1 inside the dissipation range $[O(l_K^3)]$ where essentially viscous flow prevails. However, the volume of the dissipation range is by far the smaller than inertial subrange $[O(l_0^3)]$, we may dispense with potential flow solution (44) to be integrated over its own region. Thus integral (59) with a small spheroidal regions excepted is to be carried out. (See Fig. 1.) Then, the integral is shown to be converted into a double integral as follows

$$P_{11}(k_1) = \frac{2}{(2\pi l)^3} \int_0^\infty d(\sigma^2) Q(\sigma^2, k_1)^2 \tag{60}$$

with $Q(\sigma^2, k_1)$ defined by

$$Q(\sigma^2, k_1) = \int_{s_1^\dagger}^\infty \sin k_1 s_1 \frac{\partial^2 \phi^{(0)}}{\partial s_1^2} ds_1 \tag{61}$$

where boundary contour $s_1 = s_1^\dagger(\sigma^2)$ is given by.

$$s_1^\dagger(\sigma^2) = s_0(1+\delta)\left[1 - \frac{\sigma^2}{s_0^2(2\delta + \delta^2)}\right] \tag{62}$$

The 1D power spectrum calculated is shown in Fig.3. Parameter δ corresponds to the slenderness ratio of the worm as detected by the direct numerical simulation, which is estimated as the order of $O(10N)^{-1}$, with N a number of the order of unity to several. For a certain range of this parameter, the spectrum shows $k_1^{-5/3}$ dependence, then with increase in δ it transits to k_1^{-2} for $\delta >> 1$, where the pair of dipoles are regarded as a quadrupole asymptotically.

7. Conclusions

6D Navier-Stokes' equation [Eq.(22)] governing turbulence is specialized to the locally homogeneous dissipation/inertial ranges in the eddy space. This equation [Eq.(34)] has revealed existence of a (pseudo-)singular solution of shape like a volcano with a ridgeline separating the inertial subrange from the dissipation range inside which is a energy black hole of turbulence. This finding of anomalous hike of the velocity wave function ((31) or (53)) in the eddy space has enabled us to rediscover the universal form of the power spectrum from the equation of fluid dynamics as distinct from the dimensional analysis of the classical theory. In fact, an analytical solution of one-parameter family defining the ridgeline contour includes a case that is close to the consensus wave number dependence of -5/3 power law by Kolmogorov. It is yet left to be answered to eliminate the parameter dependence. It will be achieved by replacing the inviscid solution valid only for inertial subrange with a prospective viscous solution that is uniformly valid throughout dissipation and inertial regions.

8. Acknowledgment

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Figure Captions

Figure 1. Pseudo-singularity formed by a pair of dipoles in the inertial subrange of the eddy space. The shaded part represents the dissipation region.[See Eq.(63).]
Figure 2. Evolution of turbulent vortices visualized by a gigantic turbulent jet : Volcano eruption of Mt. Saint Helens[7].
Figure 3. 1D power spectrum P_{11} plotted against 1D wave number k_1 with slenderness ratio δ as the parameter. The calculated spectrum is averaged over five neighboring points to make the comparison with -5/3 or -2 power laws easier.

Fig.1

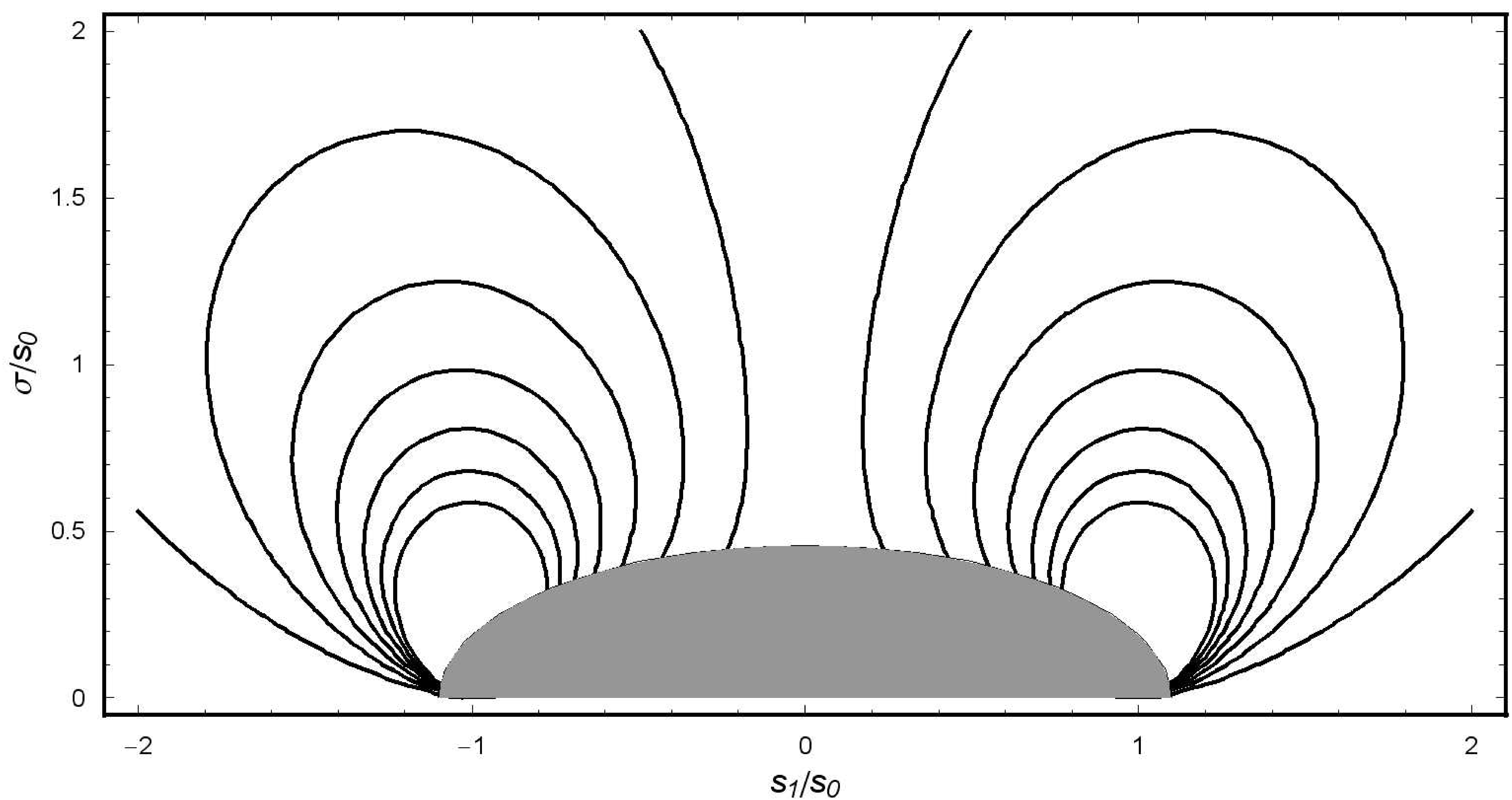

Fig. 2

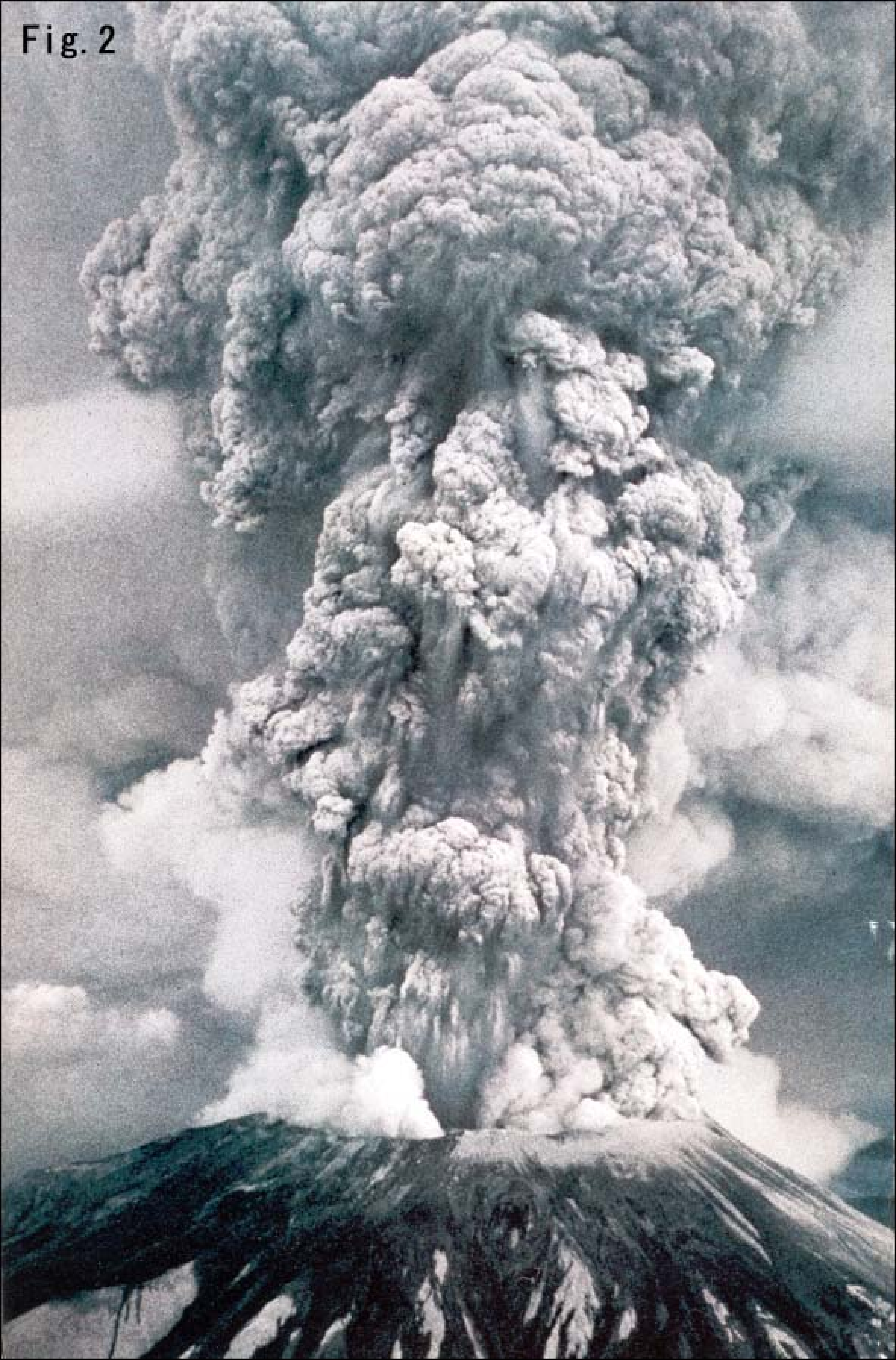

Fig.3